

# A New Kind of EH B-spline Curve With a Shape Parameter

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**ABSTRACT:** A new interpolation B-spline with two shape parameters, called EH B-spline, is presented in this paper, which is the extension of the standard cubic Hermite interpolation spline, and inherits the same properties of the standard cubic Hermite interpolation spline. Each curve segment is generated by four consecutive control points. These curves possess most optimal properties of B-spline basis. Given the fixed interpolation conditions, the shape of the proposed splines can be adjusted by changing the values of the parameters. The lower degree of the basis functions provides less computational complexity. The associated EH B-spline surfaces can exactly represent the surfaces of revolution. We can easily obtain different shapes by altering the value of shape parameter.

**KEYWORDS:** EH B-spline curve, shape parameter, blending function, EH B-spline surface, Rotational surface. Trigonometric Bézier Basis Function, Trigonometric Bézier Curve, Cubic Trigonometric Polynomial B-Spline Curve, Continuity. Introduction

## I. INTRODUCTION

The Bézier curves and surfaces form a basic tool for constructing free form curves and surfaces. Many basis-like Bézier basis are presented. Said (1989) and Goodman and Said (1991) constructed the Ball basis. Mainar *et al.* (2001) found some bases for the spaces  $\{1, t, \cos t, \sin t, \cos 2t, \sin 2t\}$ ,  $\{1, t, t^2, \cos t, \sin t\}$ , and  $\{1, t, \cos t, \sin t, t \cos t, t \sin t\}$ . Chen and Wang (2003) gave the C-Bézier basis in the space  $\{1, t, t_2, \dots, t_{n-2}, \sin t, \cos t\}$ . Wang and Wang (2004) put forward Uniform B Spline with shape parameter which is extended from Uniform B Spline and possesses many excellent properties. B-splines are widely applied to the modeling of free-form curves and surfaces. However, there are several limitations of the B-spline model, which restrict its applications. To overcome the drawbacks of B-splines, many attempts aiming at new modeling methods have been explored in recent years. In order to enhance the flexibility of B-spline models, some researchers have suggested many types of curves with shape parameters incorporated into the basis functions. Based on quartic blending functions, piecewise polynomial curves with a shape parameter were constructed by Han and Liu [9]. Wang et al. presented piecewise algebraic and trigonometric uniform B-spline curves with a shape parameter of degree  $k$  ( $k \leq 2$ ) [10,11]. However, a common problem of these improved designs/models is that they are unable to achieve local changes on individual elements because the shape parameters take effect globally which implies that they can only adjust the global shape of the curves.

In other words, they are lack of the local shape adjustability which is especially important. Researchers have introduced extended Bézier and B-spline models with some parameters in [15–17]. Xu and Wang in [17] presented two classes of polynomial blending functions with local shape parameters  $\lambda_i$  and proposed a method of generating piecewise polynomial curves. By changing the values of the local shape parameters  $\lambda_i$ , the shape of the curves can be manipulated locally to a certain extent. However, since only one kind of local shape parameters  $\lambda_i$  is included, the local adjustability of curves is limited. Spline interpolation is a useful and powerful tool for curves and surfaces modeling. Standard cubic Hermite spline is one of those interpolation functions. But, for the given interpolation condition, the cubic Hermite interpolation spline is fixed; that is to say, the shape of the interpolation curve or surface is fixed for the given interpolation data [1–6]. Since the interpolation function is unique for the given interpolation data, to modify the shape of the interpolation curve to approximate the given curve seems to be impossible and it is contradictory to the

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 8, August 2016

uniqueness of the interpolation function for the given interpolation data. For the given interpolation condition, how to improve the approximation accuracy of the interpolation spline is an important problem in the computer aided geometric design. In recent years, many authors have presented some new method to modify the shape of the interpolation curve to satisfy the industrial product design with several kinds of new interpolation splines with parameters [7–20]. These new splines all have similar properties of the standard cubic Hermite spline. For example, for the given interpolation data, if interpolation interval approaches zero, theoretically speaking, these splines can approximate the given curve and surface well. However, there exists a problem, in the process of the actual calculation; the amount of computation will increase dramatically if the length of the interpolating intervals decreases. On the other hand, the approximation accuracy of these new splines may not be better than the standard cubic Hermite spline. In [7–17], many rational form interpolation splines with multiple parameters were presented. For the given interpolation data, the change of the parameters causes the change of the interpolation curve. Nevertheless, the computation of the splines with multiple parameters is very complicated. Several kinds of rational splines with a single parameter were presented in the papers [18, 19], which is simple to compute, but its approximation accuracy is not good for the given curves and surfaces. In general, polynomial-form splines are suitable to be used to design and compute. In [20], a polynomial-form spline, called quartic Hermite spline with single parameter, is presented as the extension of the standard Hermite spline. The quartic spline has a simple form, and its approximation rate to the given curves and surfaces is not high.

In this paper, a class of new quartic splines with two parameters is developed, which is the extension of the standard cubic Hermite interpolation spline and inherits the same properties of the standard cubic Hermite interpolation spline. For the given interpolation conditions, the shape of the proposed splines can be adjusted by changing the values of the parameters. Furthermore, the introduced splines could approximate to the interpolated functions better than the standard cubic Hermite interpolation splines and the quartic Hermite interpolation splines with single parameter. The rest of this paper is organized as follows. In Section 2, the EH B-spline basis functions with two shape parameters are constructed, and some properties of the basis are given. In Section 3, the corresponding EH B-spline curve is defined, the properties of the EH B-spline curve are discussed. In Section 4 effects of the shape parameter on EH B-spline curve are studied. In Section 5, joining of two EH B-spline curve are discussed. In Section 6 the corresponding EH B-spline surface with two shape parameter is presented. Conclusion is given in Section 7.

## II. THE EH BASIS FUNCTIONS WITH TWO SHAPE PARAMETERS

### The Standard Cubic Hermite Spline and Its Basis Functions

Generally, for  $t \in [0,1]$  the following four basis functions

$$\alpha_0(t) = 1 - 3t^2 + 2t^3;$$

$$\alpha_1(t) = 3t^2 - 2t^3;$$

$$\beta_0(t) = t - 2t^2 + t^3;$$

$$\beta_1(t) = -t^2 + t^3;$$

are called the standard cubic Hermite bases. These bases satisfy

$$\alpha_0(0) = \alpha_1(1) = 1; \alpha_0(1) = \alpha_1(0) = 0;$$

$$\alpha_1(0) = \alpha_0(1) = 0; \alpha_0'(0) = \alpha_0'(1) = 0;$$

$$\beta_0(0) = \beta_1(1) = 0; \beta_1(0) = \beta_1(1) = 0;$$

$$\beta_0'(0) = \beta_1'(1) = 1; \beta_0'(1) = \beta_1'(0) = 0;$$

$$\alpha_0(t) + \alpha_1(t) = 1; \beta_0(t) + \beta_1(1-t) = 0;$$

For given knots  $a = x_0 < x_1 < \dots < x_n = b$  and data  $\{(x_i, y_i, d_i), i = 0, 1, \dots, n\}$ ,

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 8, August 2016

where  $y_i$  and  $d_i$  are the values of the function value and the first-order derivative value of function being Interpolated,

$$\text{let } h_i = x_{i+1} - x_i; t = \frac{x_{i+1} - x_i}{h_i};$$

then the standard cubic Hermite spline in the interval  $[x_i, x_{i+1}]$  can be defined as follows:

$$H_i(x) = \alpha_0(t)y_i + \alpha_1(t)y_{i+1} + \beta_0(t)h_i d_i + \beta_1(t)h_{i+1}d_{i+1};$$

For  $i = 0, 1, \dots, n$ ;

$$H_i(x_i) = y_i, H_i(x_{i+1}) = y_{i+1},$$

$$H_i'(x_i) = d_i, H_i'(x_{i+1}) = d_{i+1},$$

The standard cubic Hermite spline is  $C^1$  continuous. However, if interpolation data is given, the shape and approximation of the spline cannot be changed.

### III. THE EH SPLINE CURVE TEXT INPAINTING

#### III(i) The Basis Functions of the EH Interpolation Spline

**Definition:** For arbitrary real number  $\lambda$  and  $\mu$ , and  $0 \leq t \leq 1$ , the following four functions with parameters  $\lambda$  and  $\mu$ ,

$$\begin{aligned} b_{0,3}(t) &= 1 + (\lambda - 3)t^2 + 2(1 - \lambda)t^3 + \lambda t^4; \\ b_{1,3}(t) &= (3 - \lambda)t^2 + 2(\lambda - 1)t^3 - \lambda t^4; \\ b_{2,3}(t) &= t + (\mu - 2)t^2 + (1 - 2\mu)t^3 + \mu t^4; \\ b_{3,3}(t) &= -(\mu + 1)t^2 + (1 + 2\mu)t^3 - \mu t^4; \end{aligned} \tag{1}$$

are called basis functions of the EH interpolation spline, briefly EH bases.

The EH bases are the extension of the standard cubic Hermite bases. When  $\lambda = \mu = 0$ , the bases are the standard cubic Hermite bases. The bases have the similar properties of the standard cubic Hermite bases.

#### Properties of EH B-spline basis functions

(a) **Terminal Properties:** At the endpoints, we know that

$$\begin{aligned} b_{0,3}(0) = b_{1,3}(1) = 1; & & b_{2,3}(0) = b_{2,3}(1) = 0; \\ b_{0,3}(1) = b_{1,3}(0) = 0; & & b_{3,3}(0) = b_{3,3}(1) = 0; \\ b'_{0,3}(0) = b'_{0,3}(1) = 0; & \text{and} & b'_{2,3}(0) = b'_{3,3}(1) = 1; \\ b'_{1,3}(0) = b'_{1,3}(1) = 0; & & b'_{2,3}(1) = b'_{3,3}(0) = 0; \end{aligned}$$

and

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 8, August 2016

$$b_{0,3}(t) + b_{1,3}(t) = 1;$$

$$b_{2,3}(t) + b_{3,3}(1-t) = 1;$$

(b) **Non-negativity:** The EH B-spline basis are non-negative on the interval [0,1]

$$b_{i,3}(t) \geq 0, i = 0, 1, 2, 3; t \in [0, 1]$$

(c) **Symmetry:** The EH B-spline basis functions are symmetric, namely,

$$b_{i,3}(t) = b_{3-i,3}(1-t) ; i = 0, 1, 2, 3 ; t \in [0, 1]$$

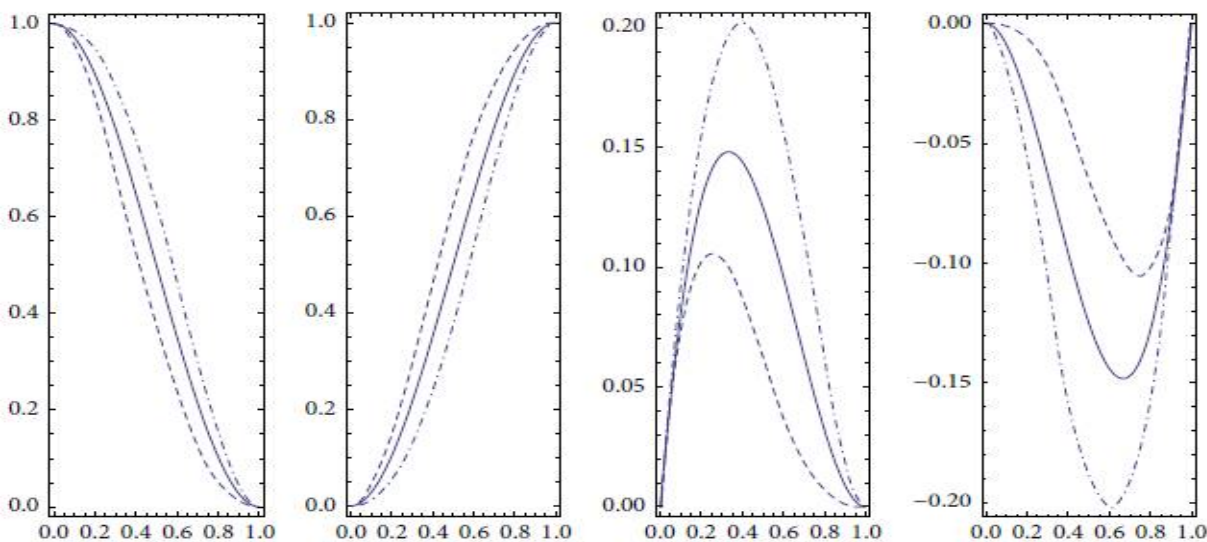
(d) **Partition of unity:** The EH B-spline basis possess weight property,

$$\sum_{i=0}^3 b_{i,3}(t) = 1; t \in [0, 1]$$

(e) **Maximum Value:** The maximum value of  $b_{i,3}(t)$  occurs at  $t = i/3, i = 0, 1, 2, 3$ .

When  $\lambda = \mu$ , the EH bases (4) are basis functions with single parameter in [20].

**Figure 1** shows the four EH bases, where the solid lines are the standard Hermite bases, the parameters  $\lambda=2, \mu=-2$  are for the dot-dash lines, and  $\lambda=-2, \mu=2$  are for dashed line.



**FIG 1 : The graph of the four basis function**

**The EH B-Spline curve with two shape parameters**

**III(ii) EH B-Spline curve:**

**Definition:EH B-Spline curve:** Given control points  $P_i (i = 0, 1, \dots, n)$  in  $R^2$  or  $R^3$  and a knot vector  $t = (t_0, t_1, \dots, t_{n+3})$ . Then the curves

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 8, August 2016

$$R_{i,3}(\lambda, \mu, t) = \sum_{j=0}^3 b_{j,3}(t)P_{i+j-3} \quad (2)$$

for  $t \in [0, 1]$ ,  $i = 1, 2, \dots, n$ . are called cubic **EH B-Spline curve segments** where  $b_{j,3}$  ( $j = 0, 1, 2, 3$ ) are EH B-spline basis functions. All the curve segments make up the piecewise cubic blending spline curves with parameters  $\lambda$  and  $\mu$ , defined by

$$R(\lambda, \mu; t) = R_{i,3}(\lambda, \mu; \frac{t - t_i}{\Delta t_i}) \quad (3)$$

For  $u \in [u_i, u_{i+1}] \subset [u_3, u_{n+1}]$ , where  $\Delta t_i = t_{i+1} - t_i$  and  $i = 3, 4, \dots, n$ .  $R(\lambda, \mu; t)$  is called EH B-spline curve with two shape parameters  $\lambda$  and  $\mu$ .

## Properties of EH B-spline curves

The EH B-Spline curves (2) have the following properties:

### (a) Terminal Properties:

$$\begin{aligned} R(\lambda, \mu; 0) &= P_0; R(\lambda, \mu; 1) = P_1 \\ R'(\lambda, \mu; 0) &= P_2; R'(\lambda, \mu; 1) = P_3 \\ R''(\lambda, \mu; 0) &= 2(\lambda - 3)(P_0 - P_1) + 2\mu(P_3 - P_4) - 2(2P_3 + P_4) \\ R''(\lambda, \mu; 1) &= (2\lambda + 6)(P_0 - P_1) + 2\mu(P_3 - P_4) + 2(P_3 + 2P_4) \end{aligned}$$

(b) **Symmetry:**  $P_0, P_1, P_2, P_3$  and  $P_3, P_2, P_1, P_0$  define the same EH B-spline curve in different parametrizations, i.e.,

$$T(\lambda, \mu; t; P_0, P_1, P_2, P_3) = T(\mu, \lambda; 1 - t; P_3, P_2, P_1, P_0) \text{ for } t \in [0, 1], \lambda, \mu \in [-1, 1].$$

(c) **Geometric invariance:** The shape of an EH B-spline curve is independent of the choice of coordinates, i.e. (2) satisfies the following two equations:

$$\begin{aligned} T(\lambda, \mu; t; P_0 + q, P_1 + q, P_2 + q, P_3 + q) &= T(\lambda, \mu; t; P_0, P_1, P_2, P_3) + q \\ T(\lambda, \mu; t; P_0 * M, P_1 * M, P_2 * M, P_3 * M) &= T(\lambda, \mu; t; P_0, P_1, P_2, P_3) * M \end{aligned}$$

for  $t \in [0, 1]$ ;  $\lambda, \mu \in [-1, 1]$  where  $q$  is arbitrary vector in  $R^2$  or  $R^3$ , and  $M$  is an arbitrary  $d \times d$  matrix,  $d = 2$  or  $3$ .

(d) **Convex hull property:** Since the EH B-spline basis functions for  $\lambda = \mu = -1, 0, 1$  have the properties of non-negativity and partition of unity, the entire EH B-spline curve segment must lie inside the control polygon spanned by  $P_{i-3}, P_{i-2}, P_{i-1}, P_i$ .

## IV. SHAPE ANALYSIS OF THE EH B-SPLINE CURVES

Given a control polygon, the shape of the usual cubic B-spline curve is fixed but for a cubic EH B-spline curve, the shape can be changed by altering the values of shape parameters  $\lambda$  and  $\mu$ .

For  $t \in [0, 1]$ , we rewrite (2) as follows:

## International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 8, August 2016

$$R_{i,3}(\lambda, \mu, t) = \sum_{j=0}^3 C_{j,3}(t)P_{i+j-3} + \lambda t^2(1+t^2-2t)(P_{i-3}-P_{i-2}) + \mu t^2(1+t^2-2t)(P_{i-1}-P_i)$$

where (4)

$$C_{0,3}(t) = (1-3t^2+2t^3); C_{1,3}(t) = (3t^2-2t^3)$$

$$C_{2,3}(t) = (t-2t^2+t^3); C_{3,3}(t) = (-t^2+t^3); \text{For } t \in [0, 1] \text{ and, } \lambda, \mu \in [-1, 1].$$

From (4) we can predict the following behaviour of the curve:

- i. When we select the same  $\lambda$  but different  $\mu$ , the EH B-spline curve segment  $R_{i,3}(\lambda, \mu, t)$  approaches to the line segment  $P_{i-3}P_{i-1}$  with the increasing (or decreasing) of shape parameters  $\mu$ .
- ii. When we select the same  $\mu$  but different  $\lambda$ , the EH B-spline curve segment  $R_{i,3}(\lambda, \mu, t)$  approaches to the line segment  $P_{i-2}P_i$  with the increasing (or decreasing) of shape parameters  $\lambda$ .
- iii. When we select different  $\lambda$  and  $\mu$  the EH B-spline curve segment  $R_{i,3}(\lambda, \mu, t)$  approaches to overall the control polygon with the concurrent increasing (or decreasing) of shape parameters  $\lambda$  and  $\mu$ .

Hence the shape parameter  $\lambda, \mu$  provides an efficient tool for obtaining various smooth shapes in geometric modelling.

### V. EHB-SPLINE CURVES JOINING

Let

$$R_1(\lambda_1, \mu_1, t) = \sum_{i=0}^3 b_{i,3}(t)P_i; t \in [0, 1], \lambda_1, \mu_1 \in [0, 1]$$

be a EH B-spline curve with two shape parameters  $\lambda_1$  and  $\mu_1$ ;

$$R_2(\lambda_2, \mu_2, t) = \sum_{i=0}^3 b_{i,3}(t)Q_i; t \in [0, 1], \lambda_2, \mu_2 \in [0, 1]$$

be another EH B-spline curve with two shape parameters  $\lambda_2$  and  $\mu_2$ .

We can then join the curve  $R_1(\lambda_1, \mu_1, t)$  to the curve  $R_2(\lambda_2, \mu_2, t)$  at  $P_3$  under the following conditions:

- (1)  $C^0$  continuity:  $P_3 = Q_0$ .
- (2)  $C^1$  continuity:  $P_3 = Q_0$  and  $P_3 - P_2 = (Q_1 - Q_0)$ .
- (3)  $C^2$  continuity:  $Q_3 = 4\lambda P_0 + (-2\lambda + 6)P_1 - 6P_2 - (2\mu + 2\lambda + 6)$

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 8, August 2016

And meeting with the  $C^1$  continuity condition.

if joining two EH B-spline curve meets with geometric continuity, so that we have under the following conditions:

(1)  $G^0$  continuity:  $P_3 = Q_0$

(2)  $G^1$  continuity:  $P_3 = Q_0$  and  $P_3 - P_2 = \omega(Q_1 - Q_0), \omega > 0$ .

(3)  $G^2$  continuity:  $P_3 = Q_0$  and  $P_3 - P_2 = \omega(Q_1 - Q_0), \omega > 0$  and  $\overline{P_1Q_2} \parallel \overline{P_2Q_1}$

## VI. THE EH B-SPLINE SURFACES

Given control points  $P_{ij}$  ( $i = 0, 1, \dots, n; j = 0, 1, \dots, m$ ) and the knot vectors  $U = (u_0, u_1, u_2, \dots, u_i, \dots, u_{n+3})$  and  $V = (v_0, v_1, v_2, \dots, v_j, \dots, v_{m+3})$ , using the tensor product method, we can construct cubic EH B-spline surface pieces

$$R_{j,3}(u, v) = \sum_{k=0}^3 \sum_{l=0}^3 b_{k,3}(\lambda_1, \mu_1; u) b_{l,3}(\lambda_2, \mu_2; v) P_{i+k-3} P_{j+l-3}$$

where  $b_{k,3}(\lambda_1, \mu_1; u)$  and  $b_{l,3}(\lambda_2, \mu_2; v)$  are the cubic EH B-spline basis functions with shape parameters  $\lambda_1, \mu_1$  and  $\lambda_2, \mu_2$  respectively. These surface pieces constitute EH B-spline surface

$$R(\lambda_1, \mu_1; \lambda_2, \mu_2; u, v) = R_{j,3}\left(\lambda_1, \mu_1; \frac{u - u_i}{\Delta u_i}, \lambda_2, \mu_2; \frac{v - v_j}{\Delta v_j}\right) \tag{5}$$

Where  $u \in [u_i, u_{i+1}] \subset [u_3, u_{n+1}]$ ,  $v \in [v_j, v_{j+1}] \subset [v_3, v_{m+1}]$ ,  $\Delta u_i = u_{i+1} - u_i$ ,  $\Delta v_j = v_{j+1} - v_j$ ;

For  $i = 3, 4, \dots, m, j = 3, 4, \dots, n$ . Then Equation (5) is called EH B-spline surface.

These surfaces have properties similar to the corresponding cubic B-spline curves. In addition, since the surfaces have four shape parameters, the shape of the surfaces can be adjusted from two direction (in each direction using two shape parameter), so they can more conveniently be used in the outline design.

## VII. CONCLUSION

This paper introduced a kind of EH B-spline curve with two shape parameters, which is the extension of the standard cubic Hermite interpolation spline. Each section of the curve is defined by only four control points. The shape of the proposed splines can be adjusted by changing the values of the parameters for the fixed interpolation conditions. Also, the structure of EH B-spline curves is similar to the cubic uniform B-spline curves so EH B-spline curves can be applied to CAD/CAM modelling system.

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Vol. 5, Issue 8, August 2016

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